**Yau College Math Competition 2022**

**Final Probability and Statistics**

**Individual Overall Exam Problems (Aug. 20-21, 2022)**

**Problem 1**. Let $X$ and $Y$ be independent identically distributed random variables with mean 0 and variance 1. Let $\phi(t)$ be their common characteristic function, and suppose that $X+Y$ and $X-Y$ are independent.

(1) Deduce a relation between $\phi(2t)$ and $\phi(\pm t)$.

(2) Show that $X$ and $Y$ are $N(0,1)$ random variables.

**Problem 2.** Let $\mathcal{N} = \{1, 2, 3, 4, \ldots\}$ be the set of natural numbers and $\mathcal{P} = \{2, 3, 5, 7, \ldots\}$ the set of all prime numbers. Write $a|b$ if $a$ divides $b$. Fix any real number $s > 1$, let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, and define a probability measure $P_s$ on $\mathcal{N}$ by $P_s(n) = \frac{1}{\zeta(s)} n^{-s}, n \in \mathcal{N}$. For each $p \in \mathcal{P}$, define a random variable $X_p$ on $\mathcal{N}$ by the formula $X_p(n) = \mathbf{1}_{\{p|n\}}(n), n \in \mathcal{N}$. Here $\{p|n\}$ denotes the event $\{n : p|n\} \subset \mathcal{N}$.

(1) Are the random variables $\{X_p : p \in \mathcal{P}\}$ independent under $P_s$?

(2) Give a probabilistic proof of Euler's identity $\frac{1}{\zeta(s)} = \prod_{p \in \mathcal{P}} (1 - p^{-s})$.